

Isotropy Properties of the Multi-Step Markov Symbolic Sequences

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A new object of the probability theory, the two-sided chain of symbols (introduced in Ref. arXiv:physics/0306170) is used to study isotropy properties of binary multi-step Markov chains with the long-range correlations. Established statistical correspondence between the Markov chains and certain two-sided sequences allows us to prove the isotropy properties of three classes of the Markov chains. One of them is the important class of weakly correlated additive Markov chains, which turned out to be equivalent to the additive two-sided sequences.

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I. INTRODUCTION

The problem of long-range correlated random symbolic systems (LRCS) has been under study for a long time in many areas of contemporary physics [1–6], biology [7–12], economics [8, 13, 14], linguistics [15–19], etc.

Among the ways to get a correct insight into the nature of correlations of complex dynamic systems, the use of the *multi-step Markov* chains is one of the most important because it allows constructing a random sequence with the prescribed correlated properties in the most natural way. The N -step Markov chains are characterized by the conditional probability that each symbol of the sequence takes on some definite value depending on N *previous* symbols. These chains can be easily constructed by sequential generation using the prescribed conditional probability function. The binary correlation functions of the Markov chains can be explicitly calculated in some simple cases. The concept of additive chains turned out to be very useful because it is possible to evaluate the binary correlation function of the chain via the memory function (see for the details Refs. [20–22]).

Another important reason for the study of Markov chains is their application to the various physical objects [23–25], e.g., to the Ising chains of classical spins. The problem of thermodynamical description of the Ising chains with long-range spin interaction is open even for the 1D case. However, the association of such systems with the Markov chains can shed light on the non-extensive thermodynamics of the LRCS.

The LRCS can also be modeled by another class of correlated sequences, the so-called two-sided chains introduced in Sec. II. They are characterized by the conditional probability that each symbol of the sequence takes on the definite value depending on the neighboring symbols *at both sides* of the considered symbol. An example of systems with such a property is the above-mentioned Ising chain. In Ref. [26], we proved that the two-sided sequences were statistically equivalent to the Markov chains. In this paper this equivalence is used to study the isotropy properties of the Markov chains.

The paper is organized as follows. In the first Section, we give the definitions of the Markov and two-sided chains and formulate the problem of this study. The next Section is devoted to the examination of the anisotropy properties of three classes of Markov chains: (a) the class of Markov chains that are equivalent to the two-sided sequences with symmetric conditional probability function; (b) the Markov chains with permutative conditional probability function; (c) the additive Markov chains, which are shown to be equivalent to the additive two-sided sequences.

II. GENERAL DEFINITIONS

Let us determine the N -step Markov chain. This is a sequence of random symbols, a_i , $i \in \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, possessing the following property: the probability of symbol a_i to have a certain value, under the condition that the values of all **previous** symbols are fixed, depends on the values of N previous symbols only,

$$P(a_i = a | \dots, a_{i-2}, a_{i-1}) = P_N(a_i = a | a_{i-N}, \dots, a_{i-2}, a_{i-1}). \quad (1)$$

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The Markov chain is a *homogeneous* sequence, because the conditional probability Eq. (1) does not depend explicitly on i , i.e., is independent of the position of symbols $a_{i-N}, \dots, a_{i-1}, a_i$ in the chain. It only depends on the values of $a_{i-N}, \dots, a_{i-1}, a_i$ and their positional relationship.

An important class of the random sequences is the *binary* chains where each symbol a_i can take on only two values, say, 0 and 1.

A very important subclass of the Markov chains is the *additive* binary ones. The conditional probability functions for these chains are described by the following formula:

$$P_N(a_i = 1 | a_{i-N}, \dots, a_{i-2}, a_{i-1}) = \bar{a} + \sum_{r=1}^N F(r)(a_{i-r} - \bar{a}). \quad (2)$$

Here $F(r)$, $r = 1, \dots, N$, is the *memory function* and \bar{a} is the average number of unities in the sequence (see, e.g., Ref. [21]).

A *permutative* binary N -step Markov chain is determined by the conditional probability that is independent of the order of symbols within the memory length N and depends on the number of unities among them only.

We define an *isotropic* chain as a sequence for which the probability of an arbitrary set of L sequential symbols occurring, referred to as the *L -word*, does not depend on the direction of "reading" the symbols. For example, the probabilities of 5-words 01100 and 00110 occurring are equal.

In different mathematical and physical problems, we are confronted with the sequences for which the conditional probability of symbol a_i to have a certain value depends on values of N **previous** and N **next** symbols only,

$$\begin{aligned} P(a_i = a | \dots, a_{i-2}, a_{i-1}, a_{i+1}, a_{i+2}, \dots) = \\ = P_{2,N}(a_i = a | a_{i-N}, \dots, a_{i-1}, a_{i+1}, \dots, a_{i+N}). \end{aligned} \quad (3)$$

We refer to these chains as *N -two-sided* sequences. One can define *additive* binary two-sided chains similarly to the Markov ones,

$$P_{2,N}(a_i = 1 | a_{i-N}, \dots, a_{i-1}, a_{i+1}, \dots, a_{i+N}) = \bar{a} + \sum_{\substack{r=-N \\ r \neq 0}}^N G(r)(a_{i+r} - \bar{a}). \quad (4)$$

Here $G(r)$, $r = \pm 1, \dots, \pm N$, is the *memory function of the two-sided chain*. For the same reason as above, see Eq. (1), this chain is homogeneous.

In paper [26], the equivalence of the two-sided and the Markov chains was proved. We have derived the equation describing the correspondence between their conditional probabilities,

$$P_{2,N}(a_i = 1 | T_i^-, T_i^+) = \frac{\prod_{\substack{r=0 \\ a_i=1}}^N P_N(a_{i+r} | T_{i+r}^-)}{\prod_{\substack{r=0 \\ a_i=1}}^N P_N(a_{i+r} | T_{i+r}^-) + \prod_{\substack{r=0 \\ a_i=0}}^N P_N(a_{i+r} | T_{i+r}^-)}. \quad (5)$$

Here $T_j^- = (a_{j-N}, \dots, a_{j-1})$ and $T_j^+ = (a_{j+1}, \dots, a_{j+N})$ are the previous and the following N -words with respect to symbol a_j .

Thus, any Markov chain can be characterized not only by the conditional probability $P_N(a_i = a | T_i^-)$, but by the two-sided conditional probability function, $P_{2,N}(a_i = a | T_i^-, T_i^+)$, as well.

III. ISOTROPY OF THE MARKOV CHAINS

A. Two-sided chains with symmetric probability function

The definition of isotropic N -step Markov chain given in Sec. II is equivalent to the following statement: the Markov chain is isotropic if its two-sided conditional probability function is symmetrical,

$$P_{2,N}(a_i = a | T_{\rightarrow}^-, T_{\rightarrow}^+) = P_{2,N}(a_i = a | T_{\leftarrow}^+, T_{\leftarrow}^-). \quad (6)$$

Here T^- and T^+ are the previous and the following N -words with respect to symbol a . The subscript “ \rightarrow ” indicates that word T is read in the direct order, from left to right, and the subscript “ \leftarrow ” shows that word T is read in the inverse order. For example, if the binary 3-two-sided chain is isotropic, then

$$P_{2,3}(a_i = 1|011, 001) = P_{2,3}(a_i = 1|100, 110).$$

Equation (6) can serve as the second definition of the isotropy for the Markov chains.

Below we prove the equivalence of these two definitions of the isotropy. This equivalence is the main result of our paper.

Suppose that the first definition is valid and, therefore, the probability $P(T_{\rightarrow}^-, a_j = a, T_{\rightarrow}^+)$ of an arbitrary $(2N+1)$ -word occurring in the Markov chain is independent of the direction in which it is read. Then, the formula for two-sided probabilities can be rewritten as

$$P_{2,N}(a_j = a|T_{\rightarrow}^-, T_{\rightarrow}^+) = \frac{P(T_{\rightarrow}^-, a_j = a, T_{\rightarrow}^+)}{\sum_b P(T_{\rightarrow}^-, a_j = b, T_{\rightarrow}^+)} = \frac{P(T_{\leftarrow}^+, a_j = a, T_{\leftarrow}^-)}{\sum_b P(T_{\leftarrow}^+, a_j = b, T_{\leftarrow}^-)} = P_{2,N}(a_j = a|T_{\leftarrow}^+, T_{\leftarrow}^-).$$

Thus, the two-sided conditional probability function of the Markov chain is symmetrical and the chain is isotropic in accordance with the second definition.

Now, let us suppose that the second definition of the isotropy is valid, i.e., the two-sided probability of the Markov chain is symmetrical.

Then, this chain, being read in the direct and inverse orders, is statistically identical.

In other words, the probability of symbol $a_i = a$ occurring under condition that **previous** N -word T_{\rightarrow} is fixed is equal to that of the same symbol occurring under the condition that the **following** N -word T_{\leftarrow} is fixed. It means that the original chain and its copy written in the inverse order have the same Markov conditional probabilities. However, the probabilities of words of arbitrary length L are determined completely by the conditional probabilities. Hence, these probabilities are the same for both chains.

Thus, the probability of an arbitrary word of length L occurring does not depend on the direction in which it is read.

According to Eq. (5) this chain, being read in the direct and inverse orders, has the same Markov conditional probabilities. However, the probabilities of words of arbitrary length L are fully governed by the conditional probabilities. Hence, this chain is isotropic according the first definition.

In the general case, the Markov chains are anisotropic. Nevertheless, our analysis of Eq. (5) shows that all 1-step and 2-step additive Markov chains are isotropic. All **non-biased** 3-step additive Markov chains are also isotropic. The additive chains can be anisotropic for $N \geq 3$. The 3-step biased Markov chains with conditional probability

$$P(a_i = 1|a_{i-3}, a_{i-2}, a_{i-1}) = \bar{a} + \sum_{r=1}^3 f_r(a_{i-r} - \bar{a}) \quad (7)$$

and $\bar{a} \neq 1/2$ are isotropic in three exceptional cases only:

1. $f_1 = f_2$. This condition is fulfilled, e.g., for the chains with the step-wise memory function.
2. $f_1 + f_2 + f_3 = 1$. It is the degenerated case. This memory function determines the Markov chain consisting completely of unities or zeroes, since $P(a_i = 1|111) = P(a_i = 0|000) = 1$.
3. $f_3 = 0$. Actually, it is the 2-step additive Markov chain.

All additive Markov chains with small memory functions, $F(r) \propto \varepsilon \ll 1$, are isotropic in the main approximation with respect to ε . This fact will be clarified in Subsec. III C 1.

It should be noted that the two-sided chain with asymmetrical conditional probability function can be considered as the Markov chain being read in the direct order and as another Markov chain being read in the inverse order. The conditional probability functions of these chains are different. From the foregoing two conclusions can be made:

1. There are, at least, two different asymmetrical N -steps Markov chains with equal correlation functions,

$$K(r) = \overline{(a_i - \bar{a})(a_{i+r} - \bar{a})}. \quad (8)$$

2. The additive anisotropic Markov chain being read in the inverse order is the non-additive Markov chain. Otherwise we would have two additive Markov chains with different memory functions having the same correlation function. But as shown in Ref. [22] it is not feasible.

B. Permutative Markov chains

1. Isotropy of permutative Markov chains

Here we prove that the permutative binary N -step Markov chain is isotropic. Let us consider two chains, M and M' , where M is the given Markov chain, $\{a_i\}$, and M' is the chain written in the inverse order, $\{a'_i\}$, $a'_i = a_{-i}$. We refer to \vec{P}_k , or $\vec{P}(a_i = 1|T(k)) = \vec{P}(1|T(k))$, as the probability of symbol a_i in chain M to be equal to unity given the previous N -word $T(k)$ contains k unities. The probability of the N -word occurring satisfies the set of linear equations,

$$\begin{cases} P(a_1, a_2, \dots, a_N) = \sum_{a_0} \vec{P}(a_N|a_0, a_1, \dots, a_{N-1})P(a_0, a_1, \dots, a_{N-1}), \\ \sum_{a_1} \dots \sum_{a_N} P(a_1, a_2, \dots, a_N) = 1. \end{cases} \quad (9)$$

The probability $P(a_1, a_2, \dots, a_N)$ of N -word occurring is determined uniquely by Eq. (9). The solution of this set of equations depends on the total number $k = a_1 + a_2 + \dots + a_N$ of the unities in the N -word (a_1, a_2, \dots, a_N) and can be presented in the form $P(a_1, a_2, \dots, a_N) = P_k$. One can easily find the following expression for P_k ,

$$P_k = P_0 \prod_{r=1}^k \frac{\vec{P}_{r-1}}{1 - \vec{P}_r}.$$

We also refer to $\overleftarrow{P}(1|T)$ as the probability of symbol a'_i to be equal to unity under condition that the previous N -word in chain M' , T , is fixed. Now let us prove that the conditional probability functions of chains M and M' are equal to each other. In other words, $\overleftarrow{P}(1|T)$ equals to \vec{P}_k for arbitrary N -word T containing exactly k unities ($k = 1, \dots, N$) and is not dependent on the order of symbols in this word. With this purpose in mind one needs to prove that

$$\overleftarrow{P}(1|[k], 0) = \overleftarrow{P}(1|[k-1], 1) = \vec{P}_k.$$

Here $[j]$ means $(N-1)$ -word containing exactly j unities, so that $(1, [k-1])$ and $([k-1], 1)$ are the N -words containing k unities. Using the definition of the conditional probability we find:

$$\overleftarrow{P}(1|[k-1], 1) = \overleftarrow{P}(1|[k-1], 1) \frac{P_k}{P_k} = \frac{P(1, [k-1], 1)}{P_k} = \vec{P}(1|1, [k-1]) = \vec{P}_k,$$

$$\overleftarrow{P}(1|[k], 0) = \frac{P(1, [k], 0)}{P_k} = (1 - \vec{P}_{k+1}) \frac{P_{k+1}}{P_k}.$$

From set (9), one gets the relation

$$P_{k+1} = \vec{P}_{k+1} P_{k+1} + \vec{P}_k P_k.$$

So, we have proved that

$$\overleftarrow{P}(1|[k-1], 1) = \overleftarrow{P}(1|[k], 0) = \vec{P}_k.$$

Thus we can refer to \vec{P}_k as the probability of symbol in chain M' to be equal to unity under condition that the previous N -word containing exactly k unities is fixed, and the conditional probability functions for the chains M and M' are equal to each other,

$$\vec{P}_k = \overleftarrow{P}_k, \quad k \geq 1. \quad (10)$$

Hence,

$$\vec{P}_0 = \overleftarrow{P}_0 \quad (11)$$

since

$$\sum_T \vec{P}(1|T)P(T) = \sum_T \overleftarrow{P}(1|T)P(T) = 1.$$

Equations (10) and (11) do imply that the permutative binary N -step Markov chains are isotropic.

2. Two-sided probability functions of permutative Markov chains

As was mentioned above (see Eq. (5)), every Markov chain can be regarded as two-sided one. Below we examine the properties of the two-sided conditional probability function of the Markov chains with one-sided conditional probability functions possessing the property of permutability. An essential point is that this property does not provide the permutability of the two-sided conditional probability function. To demonstrate this fact we will show that in the general case the two-sided conditional probability function changes its value when two neighboring symbols 1 and 0 are transposed.

Let us prove this statement by contradiction and suppose that the two-sided conditional probability function $P(a_i = 1|T_i^-, T_i^+)$ takes on the same value for two variants of the word T_i^- . These variants of the word T_i^- only differ in the values of the neighboring symbols a_j and a_{j+1} : $a_j = 1, a_{j+1} = 0$ in the first variant and $a_j = 0, a_{j+1} = 1$ in the second one. Consider the structure of Eq. (5). Taking into account the permutability of the Markov conditional probability function one can see that N factors in all products in Eq. (5) coincide for two symbols sets under study. In the general case, only one factor, $P_N(a_{j+N+1}|T_{j+N+1}^-)$, changes its value. If $a_{j+N+1} = 1$, the coincidence of the values of $P(a_i = 1|T_i^-, T_i^+)$ for two sets of symbols under the consideration yields the following relation between the values of one-sided conditional probability function:

$$\vec{P}_k^2 = \vec{P}_{k+1} \vec{P}_{k-1} \quad (12)$$

for $k = a_{j+2} + a_{j+3} \dots + a_{j+N}$, $1 \leq k \leq N-1$. In the opposite case, at $a_{j+N+1} = 0$, the similar requirement is

$$(1 - \vec{P}_k)^2 = (1 - \vec{P}_{k+1})(1 - \vec{P}_{k-1}). \quad (13)$$

These two relations are compatible for the non-correlated chain only where \vec{P}_k is k -independent. The exception is the case of one-step binary Markov chain that always has permutative two-sided conditional probability function.

Thus, any correlated multi-step permutative Markov chain possesses the non-permutative two-sided probability function.

C. Additive weakly correlated Markov chains

The third class of the isotropic sequences represents the additive weakly correlated Markov chains. For these chains, we suppose that the memory function is small,

$$\sum_{r=1}^N |F(r)| \ll 1. \quad (14)$$

Their asymptotical isotropy is proved in Subsection III C 1.

Every function of N variables, $f(a_1, \dots, a_N)$, satisfying the evident restriction, $0 \leq f(a_1, \dots, a_N) \leq 1$, can be thought of the conditional probability function $P(a_i = 1|a_{i-N}, \dots, a_{i-2}, a_{i-1})$ of some Markov chain. Yet not every function of $2N$ variables, even if restricted by the similar condition, is the conditional probability function of some two-sided chain. It follows from Eq. (5), that an arbitrary binary Markov chain is determined by 2^N parameters, i.e. the number of all possible sets of arguments in the conditional probability function. Hence, a two-sided chain equivalent to this Markov one, is also determined by 2^N parameters. Nevertheless, the two-sided conditional probability function $P(a_i = 1|T_i^-, T_i^+)$ is formally governed by 2^{2N} parameters, i.e. the number of all possible **previous**, T_i^- , and **following**, T_i^+ , N -words. So, not every function of $2N$ arguments can play a role of some two-sided conditional probability function. The example is an additive two-sided chain with small memory function $G(r)$ in Eq. (4). In this case, $G(r)$ has to be asymptotically even. This fact is proven in the Subsection III C 2.

1. Isotropy of additive weakly correlated Markov chains

In order to find the conditional probability function, $P(a_i = 1|T_i^-, T_i^+)$, of the weakly correlated additive Markov chain one has to substitute the probability Eq. (2) into Eq. (5), and retain the terms of the zeroth and first orders in $F(r)$. The obtained two-sided conditional probability function takes the form of Eq. (4) with even memory function: $G(r) = G(-r) = F(r)$. So, the additive weakly correlated Markov chains are asymptotically isotropic.

2. Restriction on the class of the memory functions of weakly correlated additive two-sided chains

In this subsection, we show that the weakly correlated additive two-sided chain is asymptotically isotropic, i.e. the two-sided memory function $G(r)$ is necessarily even. To this end we consider arbitrary two-sided additive chain and find its one-sided conditional probability function, $P(a_i = 1|T_i^-)$. We will prove that this function is reduced to the additive form, Eq. (2), with the memory function $F(r) = G(-r)$, and, therefore, the chain under consideration is asymptotically isotropic.

Let us examine the additive two-sided chain (not obligatory isotropic) and find its one-sided conditional probability function. In the general case the problem is reduced to solving the set of 2^N non-linear equations, Eq. (5), written for different sequences of symbols in word T_i^- . We consider weakly correlated additive two-sided chain subjected to the restriction $\sum_{r=1}^N (|G(-r)| + |G(r)|) \ll 1$. Its one-sided conditional probability function can be presented in more convenient form:

$$P(a_i|T_i^-) = 1 - a_i + (2a_i - 1)(\bar{a} + \varphi(T_i^-)), \quad (15)$$

with function $\varphi(T_i^-)$ to be determined. The evident equation $P(a_i = 1|T_i^-) = 1 - P(a_i = 0|T_i^-)$ is fulfilled for Eq. (15). If $G(r)$ tends to zero, the probability $P(a_i = 1|T_i^-)$ goes to \bar{a} and function φ tends to zero: $\varphi(T_i^-) \rightarrow 0$. Now, substituting the conditional probability function in the form of Eq. (15) into Eq. (5) and retaining only terms of the zeroth and first orders in $\varphi(T_i^-)$, we obtain the approximate expression for the two-sided conditional probability function:

$$P(a_i = 1|T_i^-, T_i^+) \simeq \bar{a} \left(1 + (1 - \bar{a}) \left(\sum_{\substack{r=0 \\ a_i=1}}^N \psi(a_{i+r}) \varphi(T_{i+r}^-) - \sum_{\substack{r=0 \\ a_i=0}}^N \psi(a_{i+r}) \varphi(T_{i+r}^-) \right) \right). \quad (16)$$

Here we introduce a new function ψ ,

$$\psi(a_i) = \frac{a_i - \bar{a}}{\bar{a}(1 - \bar{a})} = \begin{cases} 1/(\bar{a} - 1), & a_i = 0 \\ 1/\bar{a}, & a_i = 1. \end{cases} \quad (17)$$

The two-sided conditional probability function of the chain under consideration is given by Eq. (4). So, finally, we obtain

$$\begin{aligned} & \sum_{\substack{r=0 \\ a_i=1}}^N \psi(a_{i+r}) \varphi(T_{i+r}^-) - \sum_{\substack{r=0 \\ a_i=0}}^N \psi(a_{i+r}) \varphi(T_{i+r}^-) \\ & \simeq \frac{1}{\bar{a}(1 - \bar{a})} \sum_{r=1}^N (G(r)(a_{i+r} - \bar{a}) + G(-r)(a_{i-r} - \bar{a})). \end{aligned} \quad (18)$$

Calculating the difference between two expressions presented by Eq. (18) written for $a_{i+N} = 1$ and $a_{i+N} = 0$, we get

$$\varphi(T_{i+N}^-)|_{a_i=1} - \varphi(T_{i+N}^-)|_{a_i=0} \simeq G(N). \quad (19)$$

Substitution of Eq. (19) in Eq. (18) yields,

$$\begin{aligned} & \sum_{\substack{r=0 \\ a_i=1}}^{N-1} \psi(a_{i+r}) \varphi(T_{i+r}^-) - \sum_{\substack{r=0 \\ a_i=0}}^{N-1} \psi(a_{i+r}) \varphi(T_{i+r}^-) \\ & \simeq \frac{1}{\bar{a}(1 - \bar{a})} \left(\sum_{r=1}^{N-1} (G(r)(a_{i+r} - \bar{a}) + G(-r)(a_{i-r} - \bar{a})) + G(-N)(a_{i-N} - \bar{a}) \right). \end{aligned} \quad (20)$$

Now we repeat this procedure $N-1$ times. At the first repeat we calculate the difference between two expressions (20) written for $a_{i+N-1} = 1$ and $a_{i+N-1} = 0$ and substitute the obtained result in Eq. (20), and so on. At the last repeat we obtain,

$$\varphi(T_i^-) \simeq \sum_{r=1}^N G(-r)(a_{i-r} - \bar{a}). \quad (21)$$

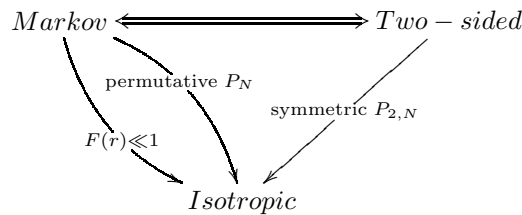
So the one-sided conditional probability function is

$$P(a_i = 1|T_i^-) \simeq \bar{a} + \sum_{r=1}^N G(-r)(a_{i-r} - \bar{a}). \quad (22)$$

As it follows from previous Subsection, Markov chains with such conditional probability functions are isotropic. Thus, we have found that the additive weakly correlated two-sided chain is asymptotically isotropic. In other words, the memory function of additive weakly correlated chain can be only even, $G(-r) = G(r)$.

IV. CONCLUSION

Thus, using the equivalence of the Markov and two-sided chains, we studied the important property of the Markov chains, their isotropy. The results of this study are shown in the scheme. Here, $A \xrightarrow{st} B$ means, that the chains from class A , restricted by the statement st , are the members of class B . The most evident fact is that the Markov chains, that possess symmetric two-sided conditional probability function, are isotropic. Another important class of the isotropic Markov chains are the sequences with the permutative conditional probability functions. One of the examples of such chains are the additive Markov chains with the step-wise memory functions, examined in details in Refs. [19, 22]. The additive weakly correlated chains are also isotropic. Such chains play a key role in the non-extensive thermodynamics of Ising chains of classical spins with long-range interaction, as well as in the literary texts and sequences of nucleotides in DNA molecules.



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